Linear Algebra (5th edition)

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Chapter 02: Linear transformations and matrices

Assignments

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1. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), and T is a function from V to W.

(a) If T is linear, then T preserves sums and scalar products.
(b) If $\mathsf{T}(x+y) = \mathsf{T}(x) + \mathsf{T}(y)$, then T is linear.
(c) T is one-to-one if and only if the only vector x such that $\mathsf{T}(x) = \mathit{0}$ is $x = \mathit{0}$.
(d) If T is linear, then $\mathsf{T}(\mathit{0}_{\mathsf{V}}) = \mathit{0}_{\mathsf{W}}$.
(e) If T is linear, then $\text{nullity}(\mathsf{T}) + \text{rank}(\mathsf{T}) = \dim(\mathsf{W})$.
(f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.
(g) If $\mathsf{T}, \mathsf{U} \colon \mathsf{V} \to \mathsf{W}$ are both linear and agree on a basis for V, then $\mathsf{T} = \mathsf{U}$.
(h) Given $x_1, x_2 \in \mathsf{V}$ and $y_1, y_2 \in \mathsf{W}$, there exists a linear transformation $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ such that $\mathsf{T}(x_1) = y_1$ and $\mathsf{T}(x_2) = y_2$.

2.1 Linear transformations, null spaces, and ranges

- **Problems**

For Exercises 2 through 6, prove that T is a linear transformation, and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

2. $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

3. $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

4. $\mathsf{T}\colon \mathsf{M}_{2\times 3}(F) \to \mathsf{M}_{2\times 2}(F)$ defined by

$$\mathsf{T}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

5. $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_3(R)$ defined by $\mathsf{T}(f(x)) = xf(x) + f'(x)$.

6. $\mathsf{T}\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\mathsf{T}(A) = \mathrm{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

2.1 Linear transformations, null spaces, and ranges

- **Problems**

10. Suppose that $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^2$ is linear, $\mathsf{T}(1,0) = (1,4)$, and $\mathsf{T}(1,1) = (2,5)$. What is $\mathsf{T}(2,3)$? Is T one-to-one?

12. Is there a linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ such that $\mathsf{T}(1,0,3) = (1,1)$ and $\mathsf{T}(-2,0,-6) = (2,1)$?

2.2 The matrix representation of a linear transformation

- **Problems**

1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $\mathsf{T}, \mathsf{U}: \mathsf{V} \to \mathsf{W}$ are linear transformations.

(a) For any scalar a, $a\mathsf{T} + \mathsf{U}$ is a linear transformation from V to W.
(b) $[\mathsf{T}]_\beta^\gamma = [\mathsf{U}]_\beta^\gamma$ implies that $\mathsf{T} = \mathsf{U}$.
(c) If $m = \dim(\mathsf{V})$ and $n = \dim(\mathsf{W})$, then $[\mathsf{T}]_\beta^\gamma$ is an $m \times n$ matrix.
(d) $[\mathsf{T} + \mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$.
(e) $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is a vector space.
(f) $\mathcal{L}(\mathsf{V}, \mathsf{W}) = \mathcal{L}(\mathsf{W}, \mathsf{V})$.

2.2 The matrix representation of a linear transformation

- **Problems**

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.

(a) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
(b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.
(c) $\mathsf{T}\colon \mathsf{R}^3 \to R$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.
(d) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3).$$

(e) $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^n$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = (a_1, a_1, \ldots, a_1)$.
(f) $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^n$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$.
(g) $\mathsf{T}\colon \mathsf{R}^n \to R$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$.

2.2 The matrix representation of a linear transformation

- **Problems**

3. Let $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ be defined by $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

10. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = \mathit{0}$. By Theorem 2.6 (p. 73), there exists a linear transformation $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[\mathsf{T}]_\beta$.