Linear Algebra (5th edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 01: Vector spaces

Assignments

Jihwan Moon

Table of Contents

- **1.1 Introduction**
- **1.2 Vector spaces**
- **1.3 Subspaces**
- **1.4 Linear combinations and systems of linear equations**
- **1.5 Linear dependence and linear independence**
- **1.6 Bases and dimension**

1.1 Introduction

- **Problems**

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$
(b) $(-3, 1, 7)$ and $(9, -3, -21)$
(c) $(5, -6, 7)$ and $(-5, 6, -7)$
(d) $(2, 0, -5)$ and $(5, 0, -2)$

2. Find the equations of the lines through the following pairs of points in space.

(a) $(3, -2, 4)$ and $(-5, 7, 1)$
(b) $(2, 4, 0)$ and $(-3, -6, 0)$
(c) $(3, 7, 2)$ and $(3, 7, -8)$
(d) $(-2, -1, 5)$ and $(3, 9, 7)$

3. Find the equations of the planes containing the following points in space.

(a) $(2, -5, -1)$, $(0, 4, 6)$, and $(-3, 7, 1)$
(b) $(3, -6, 7)$, $(-2, 0, -4)$, and $(5, -9, -2)$
(c) $(-8, 2, 0)$, $(1, 3, 0)$, and $(6, -5, 0)$
(d) $(1, 1, 1)$, $(5, 5, 5)$, and $(-6, 4, 2)$

1.2 Vector spaces

- **Problems**

1. Label the following statements as true or false.

(a) Every vector space contains a zero vector.
(b) A vector space may have more than one zero vector.
(c) In any vector space, $ax = bx$ implies that $a = b$.
(d) In any vector space, $ax = ay$ implies that $x = y$.
(e) A vector in F^n may be regarded as a matrix in $\mathsf{M}_{n\times 1}(F)$.
(f) An $m \times n$ matrix has m columns and n rows.

1.2 Vector spaces

- **Problems**

3. If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

what are M_{13}, M_{21}, and M_{22}?

4. Perform the indicated operations.

(a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

(c) $4\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

(d) $-5\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix}$

1.2 Vector spaces

- **Problems**

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over R with these operations? Justify your answer.

17. Let $\mathsf{V} = \{(a_1, a_2) \colon a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in \mathsf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

1.3 Subspaces

- **Problems**

1. Label the following statements as true or false.

(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
(b) The empty set is a subspace of every vector space.
(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
(d) The intersection of any two subsets of V is a subspace of V.
(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.
(f) The trace of a square matrix is the product of its diagonal entries.
(g) Let W be the xy-plane in R^3; that is, $\mathsf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathsf{W} = \mathsf{R}^2$.

1.3 Subspaces

- **Problems**

2. Determine the transpose of each of the matrices that follow. In addition, if the matrix is square, compute its trace.

(a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & 3 & 5 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ **(h)** $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

1.3 Subspaces

- **Problems**

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers.

(a) $\mathsf{W}_1 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $\mathsf{W}_2 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 = a_3 + 2\}$
(c) $\mathsf{W}_3 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
(d) $\mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
(e) $\mathsf{W}_5 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
(f) $\mathsf{W}_6 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

1.4 Linear combinations and systems of linear equations

- **Problems**

1. Label the following statements as true or false.

(a) The zero vector is a linear combination of any nonempty set of vectors.

(b) The span of $\varnothing$ is $\varnothing$.

(c) If S is a subset of a vector space V, then $\operatorname{span}(S)$ equals the intersection of all subspaces of V that contain S.

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

(f) Every system of linear equations has a solution.

1.4 Linear combinations and systems of linear equations

- **Problems**

2. Solve the following systems of linear equations by the method introduced in this section.

(a)
$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 \phantom{{}+5x_4} &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 2x_3 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + 2x_2 + 2x_3 \phantom{{}+5x_4} &= 2 \\ x_1 \phantom{{}+2x_2} + 8x_3 + 5x_4 &= -6 \\ x_1 + x_2 + 5x_3 + 5x_4 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= 7 \\ -x_1 \phantom{{}+2x_2} + 10x_3 - 3x_4 - 4x_5 &= -16 \\ 2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 &= 2 \\ 4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 &= 7 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 2x_2 + 6x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + x_2 - x_3 &= 15 \\ x_1 + 3x_2 + 10x_3 &= -5 \end{aligned}$$

1.4 Linear combinations and systems of linear equations

- **Problems**

3. For each of the following lists of vectors in R^3, determine whether the first vector can be expressed as a linear combination of the other two.

(a) $(-2,0,3), (1,3,0), (2,4,-1)$
(b) $(1,2,-3), (-3,2,1), (2,-1,-1)$
(c) $(3,4,1), (1,-2,1), (-2,-1,1)$
(d) $(2,-1,0), (1,2,-3), (1,-3,2)$
(e) $(5,1,-5), (1,-2,-3), (-2,3,-4)$
(f) $(-2,2,2), (1,2,-1), (-3,-3,3)$

5. In each part, determine whether the given vector is in the span of S.

(a) $(2,-1,1), \quad S = \{(1,0,2), (-1,1,1)\}$
(b) $(-1,2,1), \quad S = \{(1,0,2), (-1,1,1)\}$
(c) $(-1,1,1,2), \quad S = \{(1,0,1,-1), (0,1,1,1)\}$
(d) $(2,-1,1,-3), \quad S = \{(1,0,1,-1), (0,1,1,1)\}$
(e) $-x^3+2x^2+3x+3, \quad S = \{x^3+x^2+x+1, x^2+x+1, x+1\}$
(f) $2x^3-x^2+x+3, \quad S = \{x^3+x^2+x+1, x^2+x+1, x+1\}$

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}, \quad S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

1.5 Linear dependence and linear independence

- **Problems**

1. Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.
(b) Any set containing the zero vector is linearly dependent.
(c) The empty set is linearly dependent.
(d) Subsets of linearly dependent sets are linearly dependent.
(e) Subsets of linearly independent sets are linearly independent.
(f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

2.[3] Determine whether the following sets are linearly dependent or linearly independent.

(a) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$

(b) $\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$

7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $\mathsf{M}_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

1.6 Bases and dimension

- **Problems**

1. Label the following statements as true or false.

(a) The zero vector space has no basis.
(b) Every vector space that is generated by a finite set has a basis.
(c) Every vector space has a finite basis.
(d) A vector space cannot have more than one basis.
(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(g) The dimension of $\mathsf{M}_{m\times n}(F)$ is $m+n$.
(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.
(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.
(j) Every subspace of a finite-dimensional space is finite-dimensional.
(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

1.6 Bases and dimension

- **Problems**

2. Determine which of the following sets are bases for R^3.

(a) $\{(1,0,-1),(2,5,1),(0,-4,3)\}$
(b) $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$
(c) $\{(1,2,-1),(1,0,2),(2,1,1)\}$
(d) $\{(-1,3,1),(2,-4,-3),(-3,8,2)\}$
(e) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$

5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of R^3? Justify your answer.

7. The vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$, and $u_5=(-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for R^3.

9. The vectors $u_1=(1,1,1,1)$, $u_2=(0,1,1,1)$, $u_3=(0,0,1,1)$, and $u_4=(0,0,0,1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1,a_2,a_3,a_4) in F^4 as a linear combination of u_1, u_2, u_3, and u_4.